Test 3 50 Minutes

Full Name

Solution

Student I.D.

THIS EXAMINATION PAPER INCLUDES **8** PAGES AND **8** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

INSTRUCTIONS: No aids except the standard Casio fx991 calculator are permitted.

Problem	Points	Score
1	6	
2	3	
3	12	
4	14	
5	30	
6	15	
7	10	
8	10	
Total:	100	

1. (6 points) Determine if the following statements are True or False. **Explain your answers** (NO CREDITS WILL BE OFFERED IF JUSTIFICATION IS NOT PROVIDED)

   (a) (2 points) If $f(x)$ is a continuous at $x = a$, then f is differentiable at $x = a$.

   False. The function $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable at $x = 0$.

   (b) (2 points) If c is a critical number of f, then f has a local maximum or a local minimum at c.

   False. The function $f(x) = x^3$ has a critical point $x = 0$, which is neither a local maximum or a local minimum.

   (c) (2 points) If $f(x)$ is an odd function, then $f'(x)$ is also an odd function.

   False. The function $f(x) = x^3$ is odd, however $f'(x) = 3x^2$ is an even function.

2. (3 points) Show that $\int_{-3}^{3} (x^4 + 4) \sin x \, dx = 0$ (Hint: $\sin(-x) = -\sin x$)

Let $f(x) = x^4 \sin x$. Then

$$f(-x) = (-x)^4 \sin(-x) = x^4 \cdot - \sin x = -x^4 \sin x = -f(x)$$

Therefore the function $f(x)$ is odd and since the limits of integration are symmetric

$$\int_{-3}^{3} (x^4 + 4) \sin x \, dx = 0$$

3. (12 points) Compute the following limits

(a) (5 points) $\lim_{x \to 0} \frac{\cos 5x - \cos 3x}{3x^2}$

This limit is of the indeterminate form $\frac{0}{0}$. Therefore we can use the L'Hospital's Rule.

$$\begin{aligned} \lim_{x \to 0} \frac{\cos 5x - \cos 3x}{3x^2} &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{-5\sin 5x + 3\sin 3x}{6x} \\ &\overset{\text{L'H}}{=} \lim_{x \to 0} \frac{-25\cos 5x + 9\cos 3x}{6} \\ &= -\frac{16}{6} = -\frac{8}{3} \end{aligned}$$

(b) (7 points) $\lim_{x \to \infty} x - \ln x$

This limit is of the indeterminate form $\infty - \infty$. We cannot directly apply L'H rule immediately.

$$\begin{aligned} \lim_{x \to \infty} x - \ln x &= \lim_{x \to \infty} \ln(e^x) - \ln x \\ &= \lim_{x \to \infty} \ln\left(\frac{e^x}{x}\right) \end{aligned}$$

To compute this limit,

$$\lim_{x \to \infty} \frac{e^x}{x} \overset{\text{L'H}}{=} \lim_{x \to \infty} \frac{e^x}{1} = \infty$$

Since $\lim_{x \to \infty} \ln x = \infty$, ln is continuous on it's domain, we can conclude that

$$\lim_{x \to \infty} x - \ln x = \infty$$

Refer to class notes: We solved the same problem in class.

4. (14 points) Compute the following derivatives

(a) (7 points) $y = \dfrac{(2x+2)^6 \cdot \cos^2 x}{\sqrt[3]{(2x-1)}}$

Use logarithmic derivatives. Express your answer in terms of x only. (Do not try to fully simplify)

Take logarithm of both sides,

$$\ln y = \ln\left(\frac{(2x+2)^6 \cdot \cos^2 x}{\sqrt[3]{(2x-1)}}\right) = 6\ln(2x+2) - 2\ln(\cos x) - \frac{1}{3}\ln(2x-1).$$

Taking derivatives of both sides, we obtain

$$\frac{y'}{y} = 6 \cdot \frac{2}{2x+2} - 2 \cdot \frac{\sin x}{\cos x} - \frac{1}{3} \cdot \frac{2}{2x-1}$$
$$\frac{y'}{y} = \frac{6}{x+1} - 2\tan x - \frac{2(2x-1)}{3}$$

Now multiplying both sides by y, we obtain

$$\begin{aligned} y' &= \left(\frac{6}{x+1} - 2\tan x - \frac{2(2x-1)}{3}\right) \cdot y \\ &= \left(\frac{6}{x+1} - 2\tan x - \frac{2(2x-1)}{3}\right) \cdot \frac{(2x+2)^6 \cdot \cos^2 x}{\sqrt[3]{(2x-1)}} \end{aligned}$$

(b) (7 points) $g(x) = \displaystyle\int_{x^2}^{0} \frac{t^2+1}{t^3+1}\,dt$

In order to compute the derivative of the above function, we will need to apply the fundamental theorem of Calculus

$$g(x) = \int_{x^2}^{0} \frac{t^2+1}{t^3+1}\,dt = -\int_{0}^{x^2} \frac{t^2+1}{t^3+1}\,dt$$

Then,

$$g'(x) = -\frac{(x^2)^2+1}{(x^2)^3+1} \cdot 2x = -\frac{x^4+1}{x^6+1} \cdot 2x$$

5. (30 points) Compute the following integrals

   (a) (15 points) $\int \frac{\sin(\sqrt{\theta})}{2}\, d\theta$ (Hint: Start by making a substitution. Use the variable s for substituion)

Let $s = \sqrt{\theta}$. Then,

$$\frac{ds}{d\theta} = \frac{1}{2\sqrt{\theta}} \implies d\theta = 2\sqrt{\theta} ds$$

Then,

$$\int \frac{\sin(\sqrt{\theta})}{2}\, d\theta = \int \frac{\sin s}{2} 2\sqrt{\theta} ds = \int \sin s\, s\, ds$$

To solve the integral, we use integration by parts

$$\begin{aligned} u = s \quad &; \quad du = ds \\ dv = \sin s; \quad &\quad v = -\cos s \end{aligned}$$

Then,

$$\begin{aligned} \int \sin s\, s\, ds &= s \cdot -\cos s - \int -\cos s\, ds \\ &= -s\cos s + \sin s + C \\ &= -\sqrt{\theta}\cos(\theta) + \sin(\sqrt{\theta}) + C \end{aligned}$$

(b) (7 points) $\displaystyle\int \frac{1}{(x-3)^2}\,dx$

Let $u = x - 3$. Then $du = dx$.

$$\begin{aligned}\int \frac{1}{(x-3)^2}\,dx &= \int \frac{1}{u^2}\,du\\ &= \frac{u^{-1}}{-1} + C = \frac{-1}{u} + C\\ &= \frac{-1}{x-3} + C\end{aligned}$$

(c) (8 points) Use your answer in part b) to compute

$$\int_4^{\infty} \frac{1}{(x-3)^2}\,dx$$

We can use the answer in part b) to write

$$\begin{aligned}\int_4^{\infty} \frac{1}{(x-3)^2}\,dx &= \lim_{t\to\infty} -\frac{-1}{x-3}\Big|_4^t\\ &= \lim_{t\to\infty}\left[-\frac{-1}{t-3} - \frac{-1}{4-3}\right]\\ &= \lim_{t\to\infty}\left[-\frac{-1}{t-3} + 1\right]\\ &= 1\end{aligned}$$

6. (15 points) A toy company wants to determine the best selling price for a new model airplane. The company estimates that the initial cost of designing the model airplane is \$16000 (fixed cost). The marginal cost, in dollars per unit, is given by $500 - 3.2x + 0.12x^2$, where x is the number of model airplane produced. The company wants to charge $1700 - 7x$ (in dollars) per model, for each model airplane it sells.

   (a) (8 points) Find the cost, revenue, and profit functions

   $C'(x) = 500 - 3.2x + 0.12x^2$ and $C(0) = 16000$.

   Cost function: $C(x) = 16000 + 500x - 1.6x^2 + 0.04x^3$.

   Revenue function: $R(x) = x \cdot p(x) = x(1700 - 7x) = 1700x - 7x^2$

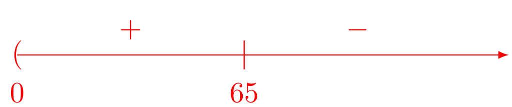
   Profit function: $P(x) = R(x) - C(x) = 1200x - 5.4x^2 - 0.04x^3 - 1600$

   (b) (7 points) Find the production level and the associated selling price that maximizes profit.

   $P'(x) = 1200 - 10.8x - 0.12x^2$
   To find the maximum value we need to compute the critical points of the function.

$$\begin{aligned} P'(x) &= 1200 - 10.8x - 0.12x^2 = 0 \\ \implies x &= \frac{10.8 \pm \sqrt{(-10.8)^2 - 4(-0.12)(1200)}}{2(-0.12)} \\ \implies x &= -154.659,\ 64.6586 \end{aligned}$$

   Since production level cannot be negative, we only have one critical point $x \approx 65$. To determine whether a maximum or minimum is attained at the critical point,
   $P'(1) > 0$ and $P'(300) < 0$



   and we see that profit function obtains a maximum value at $x = 65$ and the associated selling price that maximizes profit is

$$p(100) = 1700 - 7 \cdot 65 = \$1245$$

7. (10 points) Find $f(x)$ if $f''(x) = \sin x + \cos x$, $f'(0) = 3$ and $f(0) = 4$.

$$\begin{aligned} &f''(x) = \sin x + \cos x \implies \\ &f'(x) = -\cos x + \sin x + C \end{aligned}$$

Since $f'(0) = 3$, we get

$$\begin{aligned} &f'(0) = -\cos 0 + \sin 0 + C \implies \\ &3 = -1 + C \implies C = 4 \end{aligned}$$

Therefore $f'(x) = -\cos x + \sin x + 4$.

Then $f(x) = -\sin x - \cos x + 4x + D$. Since $f(0) = 4$,

$$\begin{aligned} &f(0) = -\sin 0 - \cos 0 + 4 \cdot 0 + D \implies \\ &4 = -1 + D \implies D = 5 \end{aligned}$$

Therefore $f(x) = -\sin x - \cos x + 4x + 5$.

8. (10 points) Find the absolute maximum and minimum value of the function

$$f(x) = x^4 - 8x^2 + 7 \text{ on } [-1, 3]$$

Since the function $f(x)$ is polynomial, it is continuous on the closed interval $[-1, 3]$. Therefore we can use the closed interval method to find absolute maximum and minimum.

$$f'(x) = 4x^3 - 16x = 4 \cdot x \cdot (x^2 - 4) = 4 \cdot x \cdot (x - 2) \cdot (x + 2)$$

The critical points of $f(x)$ are $0, 2, -2$. However since -2 is not in the interval $[-1, 3]$, we only need to evaluate the function at the endpoints $-1, 3$ and the critical points $0, 2$.

$$\begin{aligned} &f(-1) = 0 \\ &f(0) = 7 \\ &f(2) = -9 \\ &f(3) = 16 \end{aligned}$$

So the absolute maximum value of the function is 19 and absolute minimum value is -9.